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UMP Invariant Tests for a Generalized Linear Model

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For a generalized normal linear model in which the covariance matrix $\mathbf{\Sigma}$ is positive definite symmetric, UMP invariant test procedures for some kinds of linear hypotheses are derived by transforming the model by an orthogonal matrix $\mathbf{L}$, consisting of orthonormal eigenvectors of $\mathbf{\Sigma}$ as the columns vectors. Here it is assumed that $\mathbf{\Sigma}$ contains unknown elements but has a certain structure making all the elements of $\mathbf{L}$ known. A sufficient condition for this assumption is also obtained to examine whether the covariance matrix $\mathbf{\Sigma}$ has such a form.

1. INTRODUCTION

We consider a problem of testing a linear hypothesis in the following generalized linear model; for $m \geqslant 2$,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_m \end{bmatrix} = \boldsymbol{\mu} + \mathbf{e}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_m \end{bmatrix} \in V, \qquad \mathbf{e} \sim N_{mr}(\mathbf{0}, \mathbf{\Sigma}),$$

$$\mathbf{\Sigma} = \begin{bmatrix} \Sigma & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Sigma & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Sigma \end{bmatrix}, \tag{1.1}$$

where $\mathbf{Y}_i$ are independent, $\mathbf{Y}_i \sim N_r(\boldsymbol{\mu}_i, \Sigma)$, and Σ is a positive definite covariance matrix; V is a p-dimensional vector space in $\mathscr{R}^{mr}$.

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It is well known that the usual F test is optimal for linear hypotheses in the ordinary linear model, where $\mathbf{\Sigma}=\sigma^2\mathbf{I}_r$, with an identity matrix, $\mathbf{I}_r$ (see Arnold [2, Chap. 7], for example). The case in which $\mathbf{\Sigma}$ is arbitrary but known positive definite can be reduced to the ordinary linear model by the transformation, $\mathbf{Y}^*=\mathbf{\Sigma}^{-1/2}\mathbf{Y}$. When $\mathbf{\Sigma}$ is unknown, every linear hypothesis cannot be always tested by such an optimal F test. In this paper, we study a case where $\mathbf{\Sigma}$ contains unknown elements but has a certain structure which makes an optimal F test available for some linear hypothesis. We use the fact that a real symmetrix matrix can be diagonalized by an orthogonal matrix $\mathbf{L}$, which can be derived from the assumed structure of $\mathbf{\Sigma}$ that all the elements of $\mathbf{L}$ are known. By transforming the model (1.1) by such an orthogonal matrix $\mathbf{L}$, $\mathbf{Y}^*=\mathbf{L}'\mathbf{Y}$, we derive the UMP invariant test for a certain linear hypothesis. A special case of this was treated in Arnold [1].

Throughout this paper, $\mathbf{M}'$ will stand for the transpose of a matrix $\mathbf{M}$. $S(\mathbf{M})$ will denote the column space of a matrix $\mathbf{M}$. Given a vector space U in $\mathscr{R}^s$, the dimension of U will be denoted by $\dim(U)$, $U^\perp$ will stand for the orthogonal complement of U. For a subspace $W\subset U$, $U \bmod W$, i.e., $U\cap W^\perp$ will be denoted by $U|W$. For a vector $\mathbf{y}\in\mathscr{R}^s$, $P_U\mathbf{y}$ will denote the orthogonal projection of $\mathbf{y}$ on U, and we will use the same symbol P_U as the matrix representation of the orthogonal projection on U relative to the standard basis in $\mathscr{R}^s$. Finally, $S_M[\lambda_i]$ will stand for the eigenspace of a matrix $\mathbf{M}$ corresponding to the eigenvalue λ_j.

In Section 2, some remarks on the covariance matrix are given. Our main results are presented in Section 3, where a transformed model is described and we derive an UMP invariant test. A special case in which $\boldsymbol{\mu}_1=\boldsymbol{\mu}_2=\cdots=\boldsymbol{\mu}_k$ is considered in Section 4, followed by some examples.

2. Remarks on the Covariance Matrix

Let $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigenvalues of $\mathbf{\Sigma}$ with d_i being the multiplicity of the ith eigenvalue, λ_i. Since $\mathbf{\Sigma}$ is a real symmetric matrix, $\mathbf{\Sigma}$ can be diagonalized by an orthogonal matrix, whose columns are a complete set of orthonormal eigenvectors. Let $\boldsymbol{l}_1(i), \boldsymbol{l}_2(i), ..., \boldsymbol{l}_{d_i}(i)$ be a set of orthonormal eigenvectors corresponding to each eigenvalue λ_i. Then letting $\mathbf{L}_i=[\boldsymbol{l}_1(i)\,\boldsymbol{l}_2(i)\cdots\boldsymbol{l}_{d_i}(i)]$ and $\mathbf{L}=[\mathbf{L}_1\mathbf{L}_2\cdots\mathbf{L}_k]$, we have the diagonalization of $\mathbf{\Sigma}$ as $\mathbf{L}'\mathbf{\Sigma}\mathbf{L}=\mathbf{\Lambda}$, where

$$\mathbf{\Lambda}=\operatorname{diag}[\lambda_1\mathbf{I}_{d_1}, \lambda_2\mathbf{I}_{d_2}, ..., \lambda_k\mathbf{I}_{d_k}]. \tag{2.1}$$

We note that each λ_i is positive since each eigenvalue of a positive definite matrix is a positive real number.

The covariance matrix $\mathbf{\Sigma}$ of (1.1) has the same eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_k$, as Σ but each multiplicity becomes md_i, since $\det(\mathbf{\Sigma} - \lambda \mathbf{I}_{mr}) = [\det(\Sigma - \lambda \mathbf{I}_r)]^m = 0$. Then letting $\mathbf{L}_i$ consist of orthonormal eigenvectors corresponding to λ_i with multiplicity md_i and letting $\mathbf{L} = [\mathbf{L}_1 \mathbf{L}_2 \cdots \mathbf{L}_k]$, we also have $\mathbf{L}'\mathbf{\Sigma}\mathbf{L} = \mathbf{\Lambda}$, where

$$\mathbf{\Lambda} = \operatorname{diag}[\lambda_1 \mathbf{I}_{md_1}, \lambda_2 \mathbf{I}_{md_2}, ..., \lambda_k \mathbf{I}_{md_k}]. \tag{2.2}$$

For a completely arbitrary covariance matrix, we may need to know the values of its elements to obtain the diagonalizing matrix L. We assume that we can find a diagonalizing matrix depending only on the structure but not on any elements of the covariance matrix. The following condition, therefore, will be imposed on the covariance matrix in (1.1),

(M1) The covariance matrix Σ has a known structure which makes all the elements of the diagonalizing matrix L known, where the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ of Σ are unknown but their multiplicities $d_1, d_2, ..., d_k$ are known.

Thus by specifying the structure of the covariance matrix Σ, we assume that we can find the diagonalizing matrix whose elements are completely known even though Σ contains unknown elements.

The following lemma gives a means of examining this kind of structure of a covariance matrix Σ, as is easily seen.

LEMMA 2.1. *Let $h_j(\Sigma)$ $(j = 1, ..., t)$ be real-valued functions of the elements of Σ. If there exists a finite number of symmetric matrices Σ_j $(j = 1, ..., t)$ diagonalized simultaneously by an orthogonal matrix* **P** *consisting of known elements in all such that*

$$\Sigma = \sum_{j=1}^{t} h_j(\Sigma)\, \Sigma_j, \tag{2.3}$$

then Σ is also diagonalized by **P**.

A typical example of Σ derived from Lemma 2.1 is a matrix expressed as

$$\Sigma = h_1(\Sigma)\, \mathbf{I}_r + h_2(\Sigma)\, \Xi, \tag{2.4}$$

where Ξ is a symmetric matrix consisting of known constant elements in all. For instance, a tridiagonal matrix (see Ukita [7]),

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & 0 & \cdots & 0 \\ \rho & 1 & \rho & \vdots & 0 \\ 0 & \rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ 0 & 0 & \cdots & \rho & 1 \end{bmatrix}, \qquad \sigma^2 > 0, \qquad 0 < \rho < 1,$$

and the covariance matrix,

$$\Sigma=\sigma^2\begin{bmatrix}1 & \rho & \rho & \cdots & \rho\\ \rho & 1 & \rho & \vdots & \rho\\ \rho & \rho & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \rho\\ \rho & \rho & \cdots & \rho & 1\end{bmatrix}, \qquad \sigma^2>0, \qquad 0<\rho<1,$$

discussed in a repeated measures model (see Arnold [2, Chap. 14]) are expressed as in the form (2.4).

3. UMP Invariant Test for Certain Linear Hypothesis

Under the condition (M1), we restrict our discussion to the model (1.1) with the covariance matrix $\boldsymbol{\Sigma}$ which contains unknown elements but has a known structure such that there is a known orthogonal matrix $\mathbf{L}=[\mathbf{L}_1 \mathbf{L}_2 \cdots \mathbf{L}_k]$ that columns of each $\mathbf{L}_i$ are orthonormal eigenvectors corresponding to the eigenvalue λ_i with multiplicity md_i such that

$$\mathbf{L}'\boldsymbol{\Sigma}\mathbf{L}=\boldsymbol{\Lambda}=\operatorname{diag}[\lambda_1\mathbf{I}_{md_1}, \lambda_2\mathbf{I}_{md_2}, ..., \lambda_k\mathbf{I}_{md_k}],$$

where $d_1, ..., d_k$ are known but $\lambda_1, \lambda_2, ..., \lambda_k$ are unknown parameters whose only restriction is that $\lambda_i>0$, $i=1, ..., k$.

We transform a model as

$$\mathbf{Y}^*=\begin{bmatrix}\mathbf{Y}_1^*\\ \mathbf{Y}_2^*\\ \vdots\\ \mathbf{Y}_k^*\end{bmatrix}=\begin{bmatrix}\mathbf{L}_1'\mathbf{Y}\\ \mathbf{L}_2'\mathbf{Y}\\ \vdots\\ \mathbf{L}_k'\mathbf{Y}\end{bmatrix}=\mathbf{L}'\mathbf{Y}, \qquad \boldsymbol{\mu}^*=\begin{bmatrix}\boldsymbol{\mu}_1^*\\ \boldsymbol{\mu}_2^*\\ \vdots\\ \boldsymbol{\mu}_k^*\end{bmatrix}=\begin{bmatrix}\mathbf{L}_1'\boldsymbol{\mu}\\ \mathbf{L}_2'\boldsymbol{\mu}\\ \vdots\\ \mathbf{L}_k'\boldsymbol{\mu}\end{bmatrix}=\mathbf{L}'\boldsymbol{\mu}\in V^*, \tag{3.1}$$

where $V^*=\mathbf{L}'V=\{\mathbf{L}'\mathbf{v} : \mathbf{v}\in V\}$.

We set $V_i^*=\mathbf{L}_i'V$, $V_i=\mathbf{L}_iV_i^*$ and define $p_i=\dim(V_i^*)$, $i=1, 2, ..., k$. Then we have the direct sum, $\sum_{i=1}^k P_{S_{\boldsymbol{\Sigma}}[\lambda_i]}=\sum_{i=1}^k \mathbf{L}_i\mathbf{L}_i'=\mathbf{I}_{mr}$ for the orthogonal projections $P_{S_{\boldsymbol{\Sigma}}[\lambda_i]}\mathbf{y}=\mathbf{L}_i\mathbf{L}_i'\mathbf{y}$, $\mathbf{y}\in\mathscr{R}^{mr}$. Equivalently, we have the orthogonal direct sum of the decomposition of the sample space, $\mathscr{R}^{mr}=S_{\boldsymbol{\Sigma}}[\lambda_1]\oplus S_{\boldsymbol{\Sigma}}[\lambda_2]\oplus\cdots\oplus S_{\boldsymbol{\Sigma}}[\lambda_k]$. We note that, for each i,

$$V_i=P_{S_{\boldsymbol{\Sigma}}[\lambda_i]}V=\mathbf{L}_i\mathbf{L}_i'V, \qquad \dim(V_i)=\dim(V_i^*)=p_i,$$

$$\|P_{V_i}\mathbf{y}\|^2=\|P_{V_i^*}\mathbf{y}_i^*\|^2, \qquad \dim(S_{\boldsymbol{\Sigma}}[\lambda_i]\mid V_i)=\dim(V_i^{*\perp})=md_i-p_i,$$

$$\|P_{S_{\boldsymbol{\Sigma}}[\lambda_i]\mid V_i}\mathbf{y}\|^2=\|P_{V_i^{*\perp}}\mathbf{y}_i^*\|^2,$$

where $\mathbf{y}_i^*=\mathbf{L}_i'\mathbf{y}$, $\mathbf{y}\in\mathscr{R}^{mr}$. V_i are seen to be orthogonal to each other.

We further impose the following conditions on the subspaces,

(A1) $\dim(V)=\sum_{i=1}^{k}\dim(V_i)$,

(A2) $\dim(S_{\boldsymbol{\Sigma}}[\lambda_i]\mid V_i])>0,\ i=1, ..., k$.

The conditions (A1) and (A2) imply that $p=\sum_{i=1}^{k} p_i$ and $md_i-p_i>0$, respectively. Clearly, $V\subset V_1\oplus\cdots\oplus V_k$ so that the condition (A1) is equivalent to that $V=V_1\oplus\cdots\oplus V_k$ and that $V^*=V_1^*\times\cdots\times V_k^*$. Hence we have $P_v\,\mathbf{y}=\sum_{i=1}^{k} P_{v_i}\mathbf{y}$, $\|P_v\,\mathbf{y}\|^2=\sum_{i=1}^{k}\|P_{v_i}\mathbf{y}\|^2$, $\mathbf{y}\in\mathscr{R}^{mr}$.

Since the transformation $\mathbf{L}'$ is invertible and known, the problems of estimation and testing hypotheses in the model (1.1) are equivalent to those of the model (3.1). Under the conditions (A1) and (A2), we now define the statistics as

$$\hat{\mu}_i^*=P_{V_i^*}\mathbf{Y}_i^*,\qquad \hat{\lambda}_i=\frac{\|P_{V_i^{*\perp}}\mathbf{Y}_i^*\|^2}{md_i-p_i}\qquad (i=1, 2, ..., k).\tag{3.2}$$

We briefly summarize the properties of the statistics below (the argument closely follows Arnold [2]). First, from the definition (3.1) it immediately follows that

Lemma 3.1. *$\mathbf{Y}_1^*, \mathbf{Y}_2^*, ..., \mathbf{Y}_k^*$ are independent, and $\mathbf{Y}_i^*\sim N_{md_i}(\boldsymbol{\mu}_i^*, \lambda_i\mathbf{I}_{md_i})$, $i=1, ..., k$.*

Lemma 3.2. *If the conditions* (A1) and (A2) hold, then the statistics $(\hat{\boldsymbol{\mu}}_1^*, \hat{\boldsymbol{\mu}}_2^*, ..., \hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_1, ..., \hat{\lambda}_k)$ are jointly complete sufficient for the model (3.1).

Proof. Let A_i^* denote an orthonormal basis matrix for the V_i^*, and define

$$h_i(\mu_i^*, \lambda_i)=\frac{1}{(2\pi\lambda_i)^{md_i/2}}\exp(-\|\mu_i^*\|^2/2\lambda_i),$$

$$Q_i(\boldsymbol{\mu}_i^*, \lambda_i)=\begin{bmatrix}-\dfrac{1}{2\lambda_i}\\[2ex] \dfrac{1}{\lambda_i}A_i^{*\prime}\boldsymbol{\mu}_i^*\end{bmatrix},\qquad T_i(\hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i)=\begin{bmatrix}(md_i-p_i)\,\hat{\lambda}_i+\|\hat{\boldsymbol{\mu}}_i^*\|^2\\ A_i^{*\prime}\hat{\boldsymbol{\mu}}_i^*\end{bmatrix}.$$

Let f^* denote the joint density of $\mathbf{Y}_1^*, \mathbf{Y}_2^*, ..., \mathbf{Y}_k^*$ and f_i^* be the marginal density of $\mathbf{Y}_i^*$. Then, by Lemma 3.1,

$$\begin{aligned}f^*(\mathbf{y}^*, \boldsymbol{\mu}^*, \boldsymbol{\Lambda})&=\prod f_i^*(\mathbf{y}_i^*, \boldsymbol{\mu}_i^*, \lambda_i)\\&=\prod h_i(\boldsymbol{\mu}_i^*, \lambda_i)\exp[Q_i(\boldsymbol{\mu}_i^*, \lambda_i)'\,T_i(\hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i)].\end{aligned}\tag{3.3}$$

Sufficiency of $(\hat{\boldsymbol{\mu}}_1^*, \hat{\boldsymbol{\mu}}_2^*, ..., \hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_1, ..., \hat{\lambda}_k)$ follows from the factorization theorem and the fact that T_i are invertible functions. The completeness can be seen by noting that the range of Q_i contains an open rectangle in $\mathscr{R}^{p_i+1}$.

The distributions of the statistics can be found immediately from the definition (3.2) and Lemma 3.1,

LEMMA 3.3. *Under the conditions* (A1) *and* (A2), $\hat{\boldsymbol{\mu}}_1^*, \hat{\boldsymbol{\mu}}_2^*, ..., \hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_1, ..., \hat{\lambda}_k$ *are independent, and for* $i = 1, ..., k$,

$$\hat{\boldsymbol{\mu}}_i^* \sim N_{md_i}(\boldsymbol{\mu}_i^*, \lambda_i P_{V_i}), \; (md_i - p_i)\,\hat{\lambda}_i \sim \lambda_i \chi^2_{md_i - p_i}(0). \tag{3.4}$$

Let

$$\hat{\boldsymbol{\mu}} = \mathbf{L}[\hat{\boldsymbol{\mu}}_1^{*\prime}, \hat{\boldsymbol{\mu}}_2^{*\prime}, ..., \boldsymbol{\mu}_k^{*\prime}]', \; \hat{\boldsymbol{\Sigma}} = \mathbf{L}\hat{\boldsymbol{\Lambda}}\mathbf{L}', \tag{3.5}$$

where $\hat{\boldsymbol{\Lambda}} = \operatorname{diag}[\hat{\lambda}_1 \mathbf{I}_{md_1}, \hat{\lambda}_2 \mathbf{I}_{md_2}, ..., \hat{\lambda}_k \mathbf{I}_{md_k}]$, a diagonal matrix of onder mr.

We note that $\hat{\boldsymbol{\mu}} = P_V \mathbf{Y}$, since $\mathbf{L}[\hat{\boldsymbol{\mu}}_1^{*\prime}, \hat{\boldsymbol{\mu}}_2^{*\prime}, ..., \hat{\boldsymbol{\mu}}_k^{*\prime}]' = \mathbf{L}P_{V^*}\mathbf{Y}^* = \mathbf{L}P_{V^*}\mathbf{L}'\mathbf{Y} = P_V\mathbf{Y}$, where P_{V^*} is expressed as $P_{V^*} = \operatorname{diag}[P_{V_1^*}, P_{V_2^*}, P_{V_3^*}, ..., P_{V_k^*}]$, under the conditions (A1) and (A2). Also $\hat{\lambda}_i$ can be expressed as

$$\hat{\lambda}_i = \frac{\| P_{S_{\boldsymbol{\Sigma}}[\lambda_i] | V_i} \mathbf{Y} \|^2}{md_i - p_i} \qquad (i = 1, ..., k). \tag{3.6}$$

By Lemma 3.2 and invertibility of the transformation, we have

THEOREM 3.4. *Under the conditions* (A1) *and* (A2), $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$ *is a complete sufficient statistic for the model* (1.1). *Further, each component is the UMVU estimator of the corresponding parameter.*

For each $i = 1, ..., k$, we let U_i be a q_i-dimensional subspace of $V_i (0 \leqslant q_i < p_i)$. Putting $U_i^* = \mathbf{L}_i' U_i$, we define

$$F_{U_i} = \frac{\| P_{V_i^* | U_i^*} \mathbf{Y}_i^* \|^2}{(p_i - q_i)\,\hat{\lambda}_i} = \frac{\| P_{V_i | U_i} \mathbf{Y} \|^2}{(p_i - q_i)\,\hat{\lambda}_i} \qquad (i = 1, 2, ..., k). \tag{3.7}$$

Noting $\dim(V_i^* \mid U_i^*) = \dim(V_i \mid U_i) = p_i - q_i$, from Lemmas 3.1, 3.2, and 3.3, we have

LEMMA 3.5. *Under the conditions* (A1) *and* (A2), *and if* $p_i - q_i > 0$, *then*

$$F_{U_i} \sim F_{p_i - q_i,\, md_i - p_i}\left(\frac{\| P_{V_i^* | U_i^*} \boldsymbol{\mu}_i^* \|^2}{\lambda_i}\right), \qquad i = 1, ..., k, \tag{3.8}$$

where $F_{n,m}(\delta)$ *denotes a noncentral* F *distribution with* n *and* m *degrees of freedom and noncentral parameter* δ.

We now consider the problem of testing certain linear hypothesis. For a subspace U_i of V_i such that $p_i - q_i > 0$, with $\dim(U_i) = q_i \geqslant 0$, let $W_i = V_1 \oplus \cdots \oplus V_{i-1} \oplus U_i \oplus V_{i+1} \oplus \cdots \oplus V_k$, $i = 1, ..., k$. We define the ith linear hypothesis to be tested as

$$H_{0i}: \boldsymbol{\mu} \in W_i \text{ versus the general alternative } H_1: \boldsymbol{\mu} \in V.$$

The following theorem then indicates an UMP invariant test.

THEOREM 3.6. *Under the conditions* (A1), (A2), *and if* $p_i - q_i > 0$, *then* UMP *invariant, size* α *test of* $H_{0i}: \boldsymbol{\mu} \in W_i$ *versus* $H_1: \boldsymbol{\mu} \in V$ *is given by*

$$\phi_i(F_i) = \begin{cases} 1 & F_i > F^{\alpha}_{p_i - q_i, md_i - p_i}, \\ 0 & F_i \leqslant F^{\alpha}_{p_i - q_i, md_i - p_i}, \end{cases} \qquad F_i = \frac{\| P_{V \mid W_i} \mathbf{Y} \|^2}{(p_i - q_i)\, \hat{\lambda}_i} \qquad (i = 1, 2, ..., k), \tag{3.9}$$

where $F^{\alpha}_{n,m}$ *denotes the upper* 100α *percentile of a central F distribution with n and m degrees of freedom.*

Proof. First, it follows from the definitions of F_{u_i}, W_i, and F_i that

$$\dim(V \mid W_i) = \dim(V_i \mid U_i) = p_i - q_i,$$

$$\| P_{V \mid W_i} \boldsymbol{\mu} \|^2 = \| P_{V_i^* \mid U_i^*} \boldsymbol{\mu}_i^* \|^2, \qquad F_i = F_{u_i}.$$

Hence, for $\boldsymbol{\mu} \in V$, $F_i \sim F_{p_i - q_i, md_i - p_i}(\| P_{V \mid W_i} \boldsymbol{\mu} \|^2 / \lambda_i)$. Specially, $\| P_{V \mid W_i} \boldsymbol{\mu} \|^2 / \lambda_i = 0$ if $\boldsymbol{\mu} \in W_i$.

To show ϕ_i is UMP invariant, we consider the group G_i of transformations of the forms

$$\begin{aligned} &g(\hat{\boldsymbol{\mu}}_1^*, \hat{\lambda}_1, ..., \hat{\boldsymbol{\mu}}_{i-1}^*, \hat{\lambda}_{i-1}, \hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i, \hat{\boldsymbol{\mu}}_{i+1}^*, \hat{\lambda}_{i+1}, ..., \hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_k) \\ &\quad = (c_1 \hat{\boldsymbol{\mu}}_1^* + \mathbf{b}_1, c_1^2 \hat{\lambda}_1, ..., c_{i-1} \hat{\boldsymbol{\mu}}_{i-1}^* + \mathbf{b}_{i-1}, c_{i-1}^2 \hat{\lambda}_{i-1}, \hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i, c_{i+1} \hat{\boldsymbol{\mu}}_{i+1}^* \\ &\qquad + \mathbf{b}_{i+1}, c_{i+1}^2 \hat{\lambda}_{i+1}, ..., c_k \hat{\boldsymbol{\mu}}_k^* + \mathbf{b}_k, c_k^2 \hat{\lambda}_k), \end{aligned}$$

where $c_j > 0$, $\mathbf{b}_j \in \mathscr{R}^{md_j} (j = 1, ..., k)$. The statistic $(\hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i)$ is a maximal invariant under G_i. The testing problem reduced by the group G_i, i.e., the one only in volving the parameter $(\boldsymbol{\mu}_i^*, \lambda_i)$ and the statistic $(\hat{\boldsymbol{\mu}}_i^*, \hat{\lambda}_i)$ considered together is just that of an ordinary linear model, in which the statistic F_i can be a maximal invariant under groups suitably set. Hence the result follows from that of an ordinary linear model (see Lehmann [5, Chap. 7] or Arnold [2, Section 7.6], for example).

4. Generalized Repeated Measures Models and the Examples

In this section, we consider a case where $\mathbf{Y}_i$ has the same mean vector $\boldsymbol{\mu}$ (i.e., $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \cdots = \boldsymbol{\mu}_m = \boldsymbol{\mu}$). This model is usually known as repeated measures model in which each subject receives the same treatment of several kinds, hence observations from an individual are typically correlated.

We, for simplicity, formulate the model in the following version; for $m \geqslant 2$ and $p = r$,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_2 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_m \end{bmatrix} = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r)\,\boldsymbol{\mu} + \mathbf{e},$$

$$\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r) \equiv \begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \\ \vdots \\ \mathbf{I}_r \end{bmatrix} \equiv \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \end{bmatrix}, \quad \boldsymbol{\mu} \in \mathscr{R}^r, \quad \mathbf{e} \sim N_{mr}(\mathbf{0}, \boldsymbol{\Sigma}). \tag{4.1}$$

Here, $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_r)$ represents the standard basis in $\mathscr{R}^r$. Further, we put $\boldsymbol{\mu} = \mathbf{A}\boldsymbol{\mu}$ and $V = S(\mathbf{A})$ so that $\dim(V) = r$.

Let $\mathbf{L}_i$ denote a matrix having orthonormal eigenvectors $\boldsymbol{l}_1(i), ..., \boldsymbol{l}_{d_i}(i)$ of the matrix $\boldsymbol{\Sigma}$ associated to an eigenvalue λ_i with multiplicity d_i $(i = 1, ..., k)$ as its columns vectors. Corresponding to $\mathbf{L}_i$, a matrix L_i having orthonormal eigenvectors of the covariance matrix $\boldsymbol{\Sigma}$ associated to an eigenvalue λ_i with multiplicity md_i as its column vectors can be expressed as follows (see Ukita and Noda [6]),

$$\mathsf{L}_i = [\mathsf{L}_{i1}, \mathsf{L}_{i2}], \qquad \mathsf{L}_{i1} = m^{-1/2} \begin{bmatrix} \boldsymbol{l}_1(i) & \boldsymbol{l}_2(i) & \cdots & \boldsymbol{l}_{d_i}(i) \\ \boldsymbol{l}_1(i) & \boldsymbol{l}_2(i) & \cdots & \boldsymbol{l}_{d_i}(i) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{l}_1(i) & \boldsymbol{l}_2(i) & \cdots & \boldsymbol{l}_{d_i}(i) \end{bmatrix}. \tag{4.2}$$

Here L_{i1} is an $mr \times d_i$ submatrix of rank d_i. L_{i2}, on the other hand, is $mr \times (m-1)\,d_i$ submatrix of rank $(m-1)\,d_i$. Since $\sum_{i=1}^{k} \mathsf{L}_i \mathsf{L}_i' = \mathbf{I}_{mr}$, $V = S(\mathbf{A})$ and $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_r)$ is the standard basis in $\mathscr{R}^r$, we, then, have

$$\begin{aligned} &V_i = S(\mathsf{L}_{i1}), \qquad S_{\boldsymbol{\Sigma}}[\lambda_i] \mid V_i = S(\mathsf{L}_{i2}), \\ &\dim(V_i) = d_i, \qquad \dim(S_{\boldsymbol{\Sigma}}[\lambda_i] \mid V_i) = (m-1)\,d_i \qquad (i = 1, ..., k), \\ &V = S(\mathsf{A}) = S(\mathsf{L}_{11}) \oplus \cdots \oplus S(\mathsf{L}_{k1}), \qquad V^{\perp} = S(\mathsf{L}_{12}) \oplus \cdots \oplus S(\mathsf{L}_{k2}). \end{aligned} \tag{4.3}$$

We now define the subspace associated with a null hypothesis as

$$W(i) = V_1 \oplus \cdots \oplus V_{i-1} \oplus V_{i+1} \oplus \cdots \oplus V_k, \qquad i = 1, ..., k. \tag{4.4}$$

Then $V \mid W(i) = V_i$ $(i = 1, ..., k)$ and the conditions (A1) and (A2) are satisfied. In this case, the null hypothesis H_{0i}: $\boldsymbol{\mu} \in W(i)$ is equivalent to H_{0i}: $\mathbf{L}'_{i1}\boldsymbol{\mu} = \mathbf{0}$ $(i = 1, ..., k)$. Hence Theorem 3.6 is restated in the following way.

THEOREM 4.1. *For each* i $(i = 1, ..., k)$, $\| P_{V_i}\mathbf{Y} \|^2$ *and* $\| P_{S_{\boldsymbol{\Sigma}}[\lambda_i] \mid V_i}\mathbf{Y} \|^2$ *are independent and*

$$\| P_{S_{\boldsymbol{\Sigma}}[\lambda_i] \mid V_i}\mathbf{Y} \|^2 \sim \lambda_i \chi^2_{(m-1)d_i}(0), \; \| P_{V_i}\mathbf{Y} \|^2 \sim \lambda_i \chi^2_{d_i}(\| P_{V_i}\boldsymbol{\mu} \|^2/\lambda_i). \tag{4.5}$$

Specially, when the H_{01} holds, $\| P_{V_i}\mathbf{Y} \|^2 \sim \lambda_i \chi^2_{d_i}(0)$, and hence

$$F_i = \frac{(m-1)\| P_{V_i}\mathbf{Y} \|^2}{\| P_{S_{\boldsymbol{\Sigma}}[\lambda_i] \mid V_i}\mathbf{Y} \|^2} \sim F_{d_i, (m-1)d_i}(0). \tag{4.6}$$

Also, the UMP invariant size α test is given by

$$\phi_i(F_i) = \begin{cases} 1 & F_i > F^{\alpha}_{d_i, (m-1)d_i}, \\ 0 & F_i \leqslant F^{\alpha}_{d_i, (m-1)d_i}. \end{cases}$$

We first illustrate Theorem 4.1 in the case where $\boldsymbol{\Sigma}$ is a symmetric circulant matrix with order 4.

EXAMPLE 4.1. Let $r = 4$, $\mathbf{Y}_i$ $(i = 1, ..., m)$ have a symmetric circulant matrix as the covariance matrix,

$$\boldsymbol{\Sigma} = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_1 & 1 \end{bmatrix}, \qquad -1 < \rho_1, \rho_2 < 1, \; \rho_1 \neq \rho_2, \; \sigma^2 > 0. \tag{4.7}$$

Taking Lemma 2.1 into account, we express $\boldsymbol{\Sigma}$ as

$$\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_4 + \sigma^2 \rho_1 \mathbf{J}_4 + \sigma^2 \rho_2 \mathbf{K}_4,$$

where

$$\mathbf{J}_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{K}_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then letting

$$\mathbf{L} = [\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3] = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{L}_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{L}_2 = \frac{1}{\sqrt{4}} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{L}_3 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{4.8}$$

we can simultaneously diagonalize $\mathbf{J}_4$ and $\mathbf{K}_4$ by the orthogonal matrix $\mathbf{L}$ as

$$\mathbf{L}'\mathbf{J}_4\mathbf{L} = \left[\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & -2 & 0 \\ & 0 & 2 \end{array}\right], \qquad \mathbf{L}'\mathbf{K}_4\mathbf{L} = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

(The diagonalization of a symmetric circulant matrix of general form can be found in Brockwell and Davis [3], for example.)

This implies that the column vectors of $\mathbf{L}_i$, $(i = 1, 2, 3)$, are eigenvectors of $\mathbf{\Sigma}$ associated to the eigenvalues, $\lambda_1 = \sigma^2(1 - \rho_2)$ of multiplicity 2, and the simple eigenvalues $\lambda_2 = \sigma^2(1 - 2\rho_1 + \rho_2)$, $\lambda_3 = \sigma^2(1 + 2\rho_1 + \rho_2)$, respectively. We note that $\mathbf{L}$, as its form, depends only on the structure of $\mathbf{\Sigma}$ and has known elements in all.

Thus, it follows from Theorem 4.1 that the null hypothesis

$$\begin{aligned} &H_{01} : \mathbf{L}_1'\boldsymbol{\mu} = 0, && \text{that is, } \mu_1 - \mu_3 = 0 \text{ and } \mu_2 - \mu_4 = 0 \\ &H_{02} : \mathbf{L}_2'\boldsymbol{\mu} = 0, && \text{that is, } \mu_1 + \mu_3 = \mu_2 + \mu_4, \\ &H_{03} : \mathbf{L}_3'\boldsymbol{\mu} = 0, && \text{that is, } \mu_1 + \mu_2 + \mu_3 + \mu_4 = 0, \end{aligned} \tag{4.9}$$

against the general alternative $H_1 : \boldsymbol{\mu} \in \mathscr{R}^4$ can be tested by

$$F_i = \frac{(m-1)\| P_{V_i}\mathbf{Y} \|^2}{\| P_{S_{\mathbf{\Sigma}}[\lambda_i]|V_i}\mathbf{Y} \|^2}, \qquad i = 1, 2, 3,$$

$$\phi_1(F_1) = \begin{cases} 1 & F_1 > F^{\alpha}_{2,2(m-1)}, \\ 0 & F_1 \leqslant F^{\alpha}_{2,2(m-1)}, \end{cases} \qquad \phi_i(F_i) = \begin{cases} 1 & F_i > F^{\alpha}_{1,m-1} \\ 0 & F_i \leqslant F^{\alpha}_{1,m-1} \end{cases} \quad (i = 2, 3). \tag{4.10}$$

Under H_{01}, $F_1 \sim F_{2,2(m-1)}(0)$, and under H_{0i}, $F_i \sim F_{1,m-1}(0)$, $i = 2, 3$. Each of these is the UMP invariant size α test.

EXAMPLE 4.2. Let $r \geqslant 2$, $\mathbf{Y}_i$ $(i=1, ..., m)$ have the covariance

$$\mathbf{\Sigma}=\sigma^2\begin{bmatrix} 1-\rho(1-w_1) & \rho\sqrt{w_1}\sqrt{w_2} & \cdots & \rho\sqrt{w_1}\sqrt{w_r} \\ \rho\sqrt{w_2}\sqrt{w_1} & 1-\rho(1-w_2) & \cdots & \rho\sqrt{w_2}\sqrt{w_r} \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sqrt{w_r}\sqrt{w_1} & \cdots & \cdots & 1-\rho(1-w_r) \end{bmatrix}, \tag{4.11}$$

where $\sigma^2(>0)$, ρ $(-1<\rho<1)$ are unknown parameters and $w_1, w_2, ..., w_r$ are known constants such that $w_i>0$ $(i=1, 2, ..., r)$ and $\sum_{i=1}^{r} w_i=1$. The matrix $\mathbf{\Sigma}$ can be expressed as indicated in Lemma 2.1 as

$$\mathbf{\Sigma}=\sigma^2(1-\rho)\,\mathbf{I}_r+\sigma^2\rho\mathbf{w}\mathbf{w}',$$

where $\mathbf{w}=(\sqrt{w_1}, \sqrt{w_2}, ..., \sqrt{w_r})'$. This matrix has a simple eigenvalue $\lambda_1=\sigma^2$ and eigenvalues $\lambda_2=\sigma^2(1-\rho)$ of multiplicity $r-1$. The submatrices $\mathbf{L}_1$ and $\mathbf{L}_2$ of $\mathbf{L}(=[\mathbf{L}_1, \mathbf{L}_2])$ consist of the corresponding eigenvectors, respectively,

$$\mathbf{L}_1=\begin{bmatrix}\sqrt{w_1}\\ \sqrt{w_2}\\ \vdots \\ \sqrt{w_r}\end{bmatrix}, \quad \mathbf{L}_2=\begin{bmatrix} \left(\dfrac{w_1 w_2}{s_1 s_2}\right)^{1/2} & \left(\dfrac{w_1 w_3}{s_2 s_3}\right)^{1/2} & \cdots & \left(\dfrac{w_1 w_r}{s_{r-1} s_r}\right)^{1/2} \\ -\left(\dfrac{s_1^2}{s_1 s_2}\right)^{1/2} & \left(\dfrac{w_2 w_3}{s_2 s_3}\right)^{1/2} & \cdots & \left(\dfrac{w_2 w_r}{s_{r-1} s_r}\right)^{1/2} \\ 0 & -\left(\dfrac{s_2^2}{s_2 s_3}\right)^{1/2} & \cdots & \left(\dfrac{w_3 w_r}{s_{r-1} S_r}\right)^{1/2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -\left(\dfrac{s_{r-1}^2}{s_{r-1} s_r}\right)^{1/2} \end{bmatrix}, \tag{4.12}$$

where $s_i=\sum_{j=1}^{i} w_j$ $(i \leqslant r)$ and $s_r=1$. This comes from the fact that

$$\begin{aligned}\mathbf{L}'[\sigma^2(1-\rho)\,\mathbf{I}_r+\sigma^2\rho\mathbf{w}\mathbf{w}']\,\mathbf{L}&=\sigma^2(1-\rho)\,\mathbf{I}_r+\sigma^2\rho\boldsymbol{l}\boldsymbol{l}'\\ &=\operatorname{diag}(\sigma^2, \sigma^2(1-\rho), ..., \sigma^2(1-\rho)),\end{aligned}$$

where $\boldsymbol{l}=(1, 0, ..., 0)'$. We note that the orthogonal matrix $\mathbf{L}=[\mathbf{L}_1, \mathbf{L}_2]$, as its form depends only on the structure of $\mathbf{\Sigma}$ and has known elements in all (the matrix of the form in (4.12) was introduced by Irwin [4]).

By Theorem 4.1, the UMP invariant size α test for testing the null hypotheses,

$$\begin{aligned} &H_{01}: \mathbf{L}_1'\boldsymbol{\mu}=0, \qquad \text{that is,} \quad \sqrt{w_1}\mu_1+\sqrt{w_2}\mu_2+\cdots+\sqrt{w_r}\mu_r=0, \\ &H_{02}: \mathbf{L}_2'\boldsymbol{\mu}=0, \qquad \text{that is,} \quad \mu_1/\sqrt{w_1}=\mu_2/\sqrt{w_2}=\cdots=\mu_r/\sqrt{w_r} \end{aligned} \tag{4.13}$$

versus the general alternative $H_1 : \boldsymbol{\mu} \in \mathscr{R}^r$ is given by

$$F_1 = \frac{(m-1)\| P_{V_1}\mathbf{Y} \|^2}{\| P_{S_{\mathbf{\Sigma}}[\lambda_1] | V_1} \mathbf{Y} \|^2}, \qquad \phi_1(F_1) = \begin{cases} 1 & F_1 > F^{\alpha}_{1,m-1}, \\ 0 & F_1 \leqslant F^{\alpha}_{1,m-1} \end{cases}, \tag{4.14}$$

$$F_2 = \frac{(m-1)\| P_{V_2}\mathbf{Y} \|^2}{\| P_{S_{\mathbf{\Sigma}}[\lambda_2] | V_2} \mathbf{Y} \|^2}, \qquad \phi_2(F_2) = \begin{cases} 1 & F_2 > F^{\alpha}_{r-1,(m-1)(r-1)}, \\ 0 & F_2 \leqslant F^{\alpha}_{r-1,(m-1)(r-1)} \end{cases}. \tag{4.15}$$

Under H_{01}, $F_1 \sim F_{1,m-1}(0)$, and under H_{02}, $F_2 \sim F_{r-1,(m-1)(r-1)}(0)$.

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